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# ON THE APPLICATION OF THE UNIVERSAL LAW OF VELOCITY DEFECT TO NONEQUILIBRIUM FLOWS IN A TURBULENT BOUNDARY LAYER

**Yu. A. Lashkov*, V. V. Mikhailov, N. V. Samoilova, and A. A. Uspenskii**

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*On the basis of the results of measurements of the mean velocity profile of a 2-D turbulent boundary layer of an incompressible liquid, the previously proposed universal law of velocity defect has been refined. It has been shown that the refined relations are also applicable to nonequilibrium flows in the boundary layer. The proposed method makes it possible to determine the friction coefficient by the known values of the form parameter and the Reynolds number calculated by the displacement thickness of the layer in a wider range of these values compared to the known Ludwieg–Tillman formula.*

**Introduction.** The law of velocity defect is proved fairly rigorously for the case of plane-parallel turbulent flows of an incompressible liquid in pipes and ducts at a Reynolds number tending to infinity [1]. By contrast, the flow in a 2-D turbulent boundary layer at $\mathrm{Re} \to \infty$ only tends to a plane-parallel one and depends on the conditions at the outer boundary of the layer [2].

If the turbulence of the outside flow can be neglected, then such a condition is the distribution along the body of the dimensionless pressure gradient $\beta$ introduced by Clauser [3]:

$$\beta = \frac{\delta^*}{\tau_w}\frac{dp_w}{dx} = -\frac{\delta^*}{\varepsilon^2}\frac{d\ln u_0}{dx}, \quad \varepsilon = \sqrt{\frac{c_f}{2}} = \sqrt{\frac{\tau_w}{\rho u_0^2}} . \tag{1}$$

In [2], it has been shown that in the limit at $\mathrm{Re}_* \to \infty$ the function $\beta$ stabilizes, i.e., it no longer depends on the $\mathrm{Re}_*$ number ($\beta = \beta(x)$). For such limiting flow conditions the law of velocity defect should be valid. However, unlike the flow in pipes and ducts, the layer characteristics will depend on the parameter $\beta(x)$. In other words, in this case the flow in the layer is locally self-similar (locally equilibrium).

At real values of $\mathrm{Re}_* \le 10^6$ the layer parameters have no time to "adapt" themselves to the variable value of $\beta$. Such a flow is called nonequilibrium, and the condition of equilibrium is achieved at a constant value of $\beta$ in a fairly lengthy portion of the body surface. In [4], for the entire range of $\beta$ values, functional dependences defining the equilibrium velocity defect profile were proposed. These relations are in fact a compilation of Coles' "law of the wake" [5] and Schofield's law of velocity defect [6] valid for equilibrium flows with large positive longitudinal pressure gradients. Tests carried out by the authors with different variants of the pressure distribution, as well as the previously known experimental data, have made it possible to modernize the relations of [4–6]. The validity of the obtained relations for nonequilibrium conditions has been checked with the examples of concrete flows and is proposed as some "universal" approximation of the velocity defect profile.

**Refinement of the Coles Law of the Wake.** The law of the wake [5] is written in the form

$$\frac{u - u_0}{A\varepsilon u_0} = \ln\xi - 2\Pi\left(1 - \frac{w}{2}\right), \quad w(0) = 0, \quad w(1) = 2, \quad dw/d\xi = 0, \quad \text{where} \quad \xi = 0 \quad \text{and} \quad 1 . \tag{2}$$

In [7], it has been shown that the function $w$ is fairly well approximated by the dependence

*Deceased

Prof. N. E. Zhukovskii Central Aerohydrodynamic Institute, Zhukovskii, 140180, Moscow Region, Russia; email: n.samoilova@mail.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 81, No. 5, pp. 834–843, September–October, 2008. Original article submitted April 18, 2007; revision submitted August 30, 2007.

$$\frac{w}{2} = \xi^2 (3 - 2\xi) . \tag{3}$$

However, in this case, according to (2), the value of $du/d\xi \neq 0$ at $\xi = 1$. An analogous result is also given by the experimental dependence $w(\xi)$ presented in [5], if for the upper boundary of the layer, following Coles, the maximum point of $w(\xi)$ is taken. In this case, from the momentum conservation equation it follows that the Reynolds stress on the upper boundary of the layer is other than zero. The more exact approximation proposed in [8]

$$\frac{w}{2} = \xi^2 [3 + \omega - (2 + \omega) \xi] , \quad \omega = \frac{1}{2\Pi} \tag{4}$$

has no such limitations. Then relation (2) can be given in the form

$$\frac{u - u_0}{u_s} = \omega \left[ \ln \xi + \xi^2 (1 - \xi) \right] + \xi^2 (3 - 2\xi) - 1 , \quad u_s = \frac{A\varepsilon u_0}{\omega} . \tag{5}$$

For conditions where $\omega = O(1)$ the value of $u_s = O(\varepsilon u_0)$ ($\varepsilon u_0$ is the dynamic velocity).

**Law of Velocity Defect for Large Positive Pressure Gradients.** As was shown in [4] and will be confirmed below, with significantly increasing longitudinal pressure gradients the values of $\varepsilon \to 0$, $\omega \to 0$, $A\varepsilon \to 1$, and $u_s \to u_0$. In the limit, the logarithmic portion of the velocity profile disappears, and when $\xi \to 0$, the value of $u/u_0 \sim \xi^{1/2}$ if the wall zone of viscosity influence, where $\Delta y = \delta^* O(\mathrm{Re}_*^{-2/3})u/u_0 = O(\mathrm{Re}_*^{-1/3})$, is neglected. According to [6], at $u < u_0$ (but $u_s = O(u_0)$), a small logarithmic portion of the velocity can be replaced by a velocity jump on the wall $\Delta u = u_0 - u_s$. Then, when $\xi \to 0$,

$$\frac{u - u_0}{u_s} \to 0.47 \left( \frac{u_s}{u_0} \frac{y}{\delta^*} \right)^{1/2} - 1 . \tag{6}$$

In [6], $\delta = 2.86\delta^* u_0/u_s$ was assumed, whence

$$\frac{u - u_0}{u_s} \to 0.79\xi^{1/2} - 1 . \tag{7}$$

In the same work, in the range of $0 \le \xi \le 1$, the dependence

$$\frac{u - u_0}{u_s} = 0.4\xi^{1/2} + 0.6 \sin\left( \frac{\pi}{2} \xi \right) - 1 \tag{8}$$

was proposed. This relation contradicts (7) and does not comply with the condition $du/d\xi = 0$ at $\xi = 1$. Therefore, avoiding the above disadvantages, let us replace (8) by the relation

$$\frac{u - u_0}{u_s} = 0.794\xi^{1/2} - 0.1846\xi + 1.3842\xi^2 - 0.9936\xi^3 - 1 , \quad \delta = 2.857\delta^* \frac{u_0}{u_s} , \quad \frac{du}{d\xi} = 0 \ \text{ at } \ \xi = 1 . \tag{9}$$

**Universal Law of Velocity Defect.** We obtain the functional form of the law of velocity defect valid for small ($u_s = O(\varepsilon u_0)$) and large positive ($u_s = O(u_0)$) pressure gradients by compiling relations (5) and (9).

Let us give dependence (9) as

$$\frac{u - u_0}{u_s} = 0.794\xi^{1/2} - 0.1846\xi - 1.6158\xi^2 + 1.0064\xi^3 + \xi^2 (3 - 2\xi) - 1 .$$

Then, combining (5) and this relation, we obtain the following "universal" formula:

$$\frac{u - u_0}{u_s} = \omega \left[ \ln \xi + (1 - \xi) \xi^2 \right] + \xi^2 (3 - 2\xi) - 1 + N (0.794\xi^{1/2} - 0.1846\xi - 1.6158\xi^2 + 1.0064\xi^3) . \tag{10}$$

Here the logarithmic portion of the velocity defect rejected in (9) has been taken into account and the function $N(\omega)$ such that $N \approx 0$ at $\omega \approx O(1)$ and $N \approx 1$ at $\omega << 1$ has been introduced.

Note that (10) contains the parameter $\omega$, i.e., unlike (9), the dependence of the velocity profile on $\xi$ is two-parametric (as in the case of the flow with small pressure gradients). Let us introduce the notation

$$\varphi_1 = \ln \xi + \xi^2 (1 - \xi) , \quad \varphi_2 = \xi^2 (3 - 2\xi) - 1 ,$$
$$\varphi_3 = 0.794\xi^{1/2} - 0.1846\xi - 1.6158\xi^2 + 1.0064\xi^3 , \quad \varphi = \omega\varphi_1 + \varphi_2 + N\varphi_3 . \tag{11}$$

Thus, relation (9) is written as

$$\frac{u - u_0}{u_s} = \varphi . \tag{12}$$

Hence for the displacement thickness $\delta^*$ we get

$$\frac{\delta^*}{\delta} = -\frac{u_s}{u_0} \int_0^1 \varphi d\xi , \quad -\int_0^1 \varphi d\xi = \frac{11}{12}\omega + 0.500 - 0.150N . \tag{13}$$

As a universal scale of velocity $u_s$, we will use scale (5) introduced in [4]

$$u_s = \frac{A\varepsilon u_0}{\omega} . \tag{14}$$

According to the Coles law of the wake (and its modification (4)), this scale, if $\omega = O(1)$, $N \approx 0$, is applicable at small pressure gradients. At the same time, at large gradients ($\omega << 1$) the conditions $u_s/u_0 \sim 1$, $N \sim 1$ provide the velocity defect profile (9).

For intermediate conditions, it is necessary, using the experimental results, to select the function $N = N(\omega)$ providing the best approximation $u/u_0 = f(\xi, \omega, \varepsilon)$. At known values of $\varepsilon$, $\omega$, $N(\omega)$ the "universal" velocity profile is defined by relation (12). To calculate $\varepsilon$ and $\omega$, one has to know any two independent dimensionless parameters of the boundary layer. As was shown in [5], these can be any of the parameters $\varepsilon$, $\delta^*/\delta$, $\mathrm{Re}_*$. Naturally, the form parameter $H$ can be added to them.

Using the expressions for the momentum thickness $\theta$ and (13), for $H$ we have

$$\frac{1}{H} = 1 + \frac{u_s}{u_0} \int_0^1 \varphi^2 d\xi \Big/ \int_0^1 \varphi d\xi . \tag{15}$$

Hence the parameter $G$ introduced by Clauser [3] equals

$$G = \frac{H - 1}{H\varepsilon} = -\frac{A}{\omega} \int_0^1 \varphi^2 d\xi \Big/ \int_0^1 \varphi d\xi . \tag{17}$$

In principle, the values of $\varepsilon$ and $\omega$ can be determined by means of (13)–(15) at $H$, $\delta^*/\delta$ known from the experiment. However, such methods do not provide an acceptable accuracy, since the experimental value of $\delta$ is rather arbitrary. Therefore, to construct the velocity profile, it is more expedient to use, as initial parameters, any two of the parameters $\varepsilon$, $H$, and $\mathrm{Re}_*$.

Let us find the relation relating $\varepsilon$, $\mathrm{Re}_*$, and $\omega$, with the help of the asymptotic formula for calculating $\varepsilon$ obtained in [1, 3]:

$$\frac{1}{A\varepsilon} = \ln \frac{\delta^*}{a\varepsilon h} . \tag{17}$$

Here $a$ is the parameter specifying the logarithmic feature of the function $(u - u_0)/A\varepsilon u_0 = \varphi/\omega$ in the variable $\zeta = y\varepsilon/\delta^*$ near the wall, where $\varphi/\omega \to \ln a\zeta$ when $\zeta \to 0$. Hence, passing to the variable $y/\delta = \xi$ and taking into account that, according to (11), (12), when $\xi \to 0$ the value of $\varphi/\omega \to \ln \xi - 1/\omega$ we obtain $-\ln a = 1/\omega + \ln \zeta_\delta$, $\zeta_\delta = \delta\varepsilon/\delta^*$. For $\zeta_\delta$ from (13), (14) we have

$$\frac{\omega}{A\zeta_\delta} = -\int_0^1 \varphi d\xi \,. \tag{18}$$

The parameter $h$ entering into (17) is [1] the characteristic size of the wall zone of flow if the "wall law" is written as $u/\varepsilon u_0 = A \ln (y/h)$. Hence, taking into account that for a hydraulically smooth surface $u/\varepsilon u_0 = A \ln (y\varepsilon u_0/\nu) + B$, we have $h = \nu/(\varepsilon u_0) \exp (A/B)$ (for the case of a rough wall the value of $h$ is given in [2]).

Consequently, under the conditions being considered

$$\frac{1}{A\varepsilon} = \ln\left[\frac{\exp (B/A)}{A} A\zeta_\delta \,\mathrm{Re}_*\right] + \frac{1}{\omega}\,, \quad \frac{\exp (B/A)}{A} = 3.181 \,. \tag{19}$$

This relation fully coincides with the formula recommended for calculating the parameter $\Pi = 1/2\omega$ in [5], if we assume $N = 0$ and use for the Coles function the initial formula (3) rather than the modernized formula (4). In the case where the $\varepsilon$ value is not known, to calculate $\varepsilon$ and $\omega$, a simultaneous solution of Eqs. (15) and (19) at given $H$ and $\mathrm{Re}_*$ is required.

**Error Sources of the Method. Refinement of the Proposed Dependences.** Let us list the basic assumptions forming the basis of the above-described method for calculating the mean velocity profile: a) the flow is close to a plane-parallel one; b) the velocity profiles are described by a two-parameter function; c) the wall zone (for a smooth wall the viscous sublayer) can be neglected; d) the boundary layer is a developed turbulent layer.

It can be expected that assumption a) will be sufficiently justified in the pre-separation regions of the flow where the curvature of the streamlines is relatively large [2]. Assumption b) is obviously justified only for the case of equilibrium (quasi-equilibrium) boundary layers where $\beta$ and $\mathrm{Re}_*$ can be taken for the above parameters. For non-equilibrium conditions, indirect evidence of the two-parameter character is, for example, the widely used Ludwieg–Tillman empirical formula [10] (valid, naturally, in some limited range of $H$ and $\mathrm{Re}_*$)

$$c_f = 0.246\,(\mathrm{Re}_\theta)^{-0.268}\cdot 10^{-678H}\,, \quad \mathrm{Re}_\theta = \frac{\mathrm{Re}_*}{H}\,. \tag{20}$$

In addition, note that the Coles "law of the wake" [5] is also based on assumption b).

Assumption c) is justified asymptotically at $\mathrm{Re}_* \to \infty$ when the wall zone becomes relatively as small as desired. The influence of the size of this zone can be estimated by introducing corresponding corrections into the values of the parameters $\delta^*$ and $\theta$ which were calculated above without taking into account the viscous sublayer.

Denote the upper boundary of the linear section of the mean velocity profile in the coordinates of the law of the wall by $y^+_{\mathrm{lin}}$. Then, replacing on this section the logarithmic profile by a linear one, we obtain for the above corrections

$$\Delta\delta^* = C_1 \frac{\delta^*}{\mathrm{Re}_*}\,, \quad \Delta\theta = (-\,C_1 + C_2\varepsilon)\frac{\delta^*}{\mathrm{Re}_*}\,,$$
$$C_1 = y^+_{\mathrm{lin}}\left[A\left(1 - \frac{1}{2}\ln y^+_{\mathrm{lin}}\right) - \frac{B}{2}\right], \quad C_2 = y^+_{\mathrm{lin}}\left[-\frac{2}{3}A^2 \ln^2 y^+_{\mathrm{lin}} + \left(2A - \frac{4}{3}B\right)A \ln y^+_{\mathrm{lin}} - (A - B)^2\right]. \tag{21}$$

Taking into account that the deviation from the logarithmic law also occurs in a certain "transition" region of the profile, assume $y^+_{\mathrm{lin}} = 15$. Then at $A = 2.44$, $B = 5$ we get $C_1 = -50.5$, $C_2 = -712$. According to what has been said (the squared corrections are neglected), the refinement of the previously obtained relations is equivalent to the replacement of the parameters $\delta^*$, $\theta$, $1/H$, $\mathrm{Re}_*$, respectively, by the following expressions:

$$\delta^*\left(1+\frac{C_1}{\mathrm{Re}_*}\right),\quad \theta\left[1-(C_1-\varepsilon C_2)\frac{H}{\mathrm{Re}_*}\right],\quad \frac{1}{H}\left(1+\frac{C_1}{\mathrm{Re}_*}\right)-\frac{C_1}{\mathrm{Re}_*}+\varepsilon\frac{C_2}{\mathrm{Re}_*},\quad \mathrm{Re}_*+C_1\,. \tag{22}$$

Then at known $\mathrm{Re}_*$ and $H$ the calculation reduces to the determination of the value of $\omega$ from the equations

$$1+\frac{C_1}{\mathrm{Re}_*}-\left(1-\frac{C_1}{\mathrm{Re}_*}\right)\frac{1}{H}=\frac{u_s}{u_0}\left(\frac{C_2\omega}{A\mathrm{Re}_*}-\int_0^1\varphi^2d\xi\Big/\int_0^1\varphi d\xi\right),$$
$$\frac{u_s}{u_0}=\frac{A\varepsilon}{\omega}=\left\{\omega\ln\left[3.181\,A\zeta_\delta\,(\mathrm{Re}_*+C_1)\right]+1\right\}^{-1},\quad A\zeta_\delta=\left(-\frac{1}{\omega}\int_0^1\varphi d\xi\right)^{-1}. \tag{23}$$

The integrals entering into (23) are equal to

$$\int_0^1\varphi^2d\xi=2.023\omega^2+1.521\omega+0.3714-N\,(0.3776\omega+0.2044)+0.0310N^2\,,$$
$$\int_0^1\varphi d\xi=-\frac{11}{12}\,\omega-\frac{1}{2}+0.150N\,. \tag{24}$$

The value of $\delta$ needed for calculating $y = \xi\delta$ is found from the relation

$$\left(1+\frac{C_1}{\mathrm{Re}_*}\right)\frac{\delta^*}{\delta}=-\frac{u_s}{u_0}\int_0^1\varphi d\xi\,. \tag{25}$$

The parameter $\varepsilon$ and, consequently, the friction coefficient are determined from the second equation of (23). Relations (11) and (12) remain unaltered, and the parameter $N(\omega)$ will be given below from the condition of the best approximation of the test data.

Obviously, the proposed method will be inapplicable in the case where assumption d) is not fulfilled, in particular, relaminarization of the layer occurs.

There exist many criteria for the beginning of relaminarization [8]. However, the experimental data show [11] that these criteria are insufficiently reliable. In [8], it was noted that one of the relaminarization conditions is a downstream decrease in the value of $\mathrm{Re}_\theta$, and basing on this property of the layer we obtain the required condition of its absence.

Let us write the integral momentum equation (valid also for the laminarization of the layer) in the form

$$\frac{d\theta}{dx}=\varepsilon^2\left(1+\frac{H+2}{H}\,\beta\right). \tag{26}$$

Hence for $\mathrm{Re}_\theta$ we get

$$\frac{d\ln\mathrm{Re}_\theta}{dx}=\frac{\varepsilon^2}{\theta}\left(1+\frac{H+1}{H}\,\beta\right).$$

Thus, $d\mathrm{Re}_\theta/dx>0$ at $\beta>-H/(H+1)$ and is a forteriori greater than zero at

$$\beta>-\frac{1}{2}\,. \tag{27}$$

This conclusion is also confirmed by the result obtained in [12], where for the equilibrium turbulent boundary layer when $\mathrm{Re}\to\infty$ (when $H\to 1$) a solution does not exist if $\beta<-1/2$. Note also that in the experiments of [11, 13] relaminarization was observed in the case where the parameter $\beta$, decreasing, reached values $-2$ and $-3$ respectively.

And the value of $\mathrm{Re}_\theta$ therewith decreased by 50 [11] and 75 [13] percent. In the tests, the minimum value of $\beta$ equalled $-0.35$, which, according to (27), should provide a developed flow in the turbulent boundary layer.

**Testing Technique.** Measurements were made on the undersurface of a forward-flow low-turbulence wind tunnel with a rectangular cross-section of the working section of 350 × 500 mm and a length of the working section of about 2.5 m. The level of rms pulsations of the longitudinal velocity was about 0.4%, and in the frequency range exceeding 5 Hz about 0.06%. The velocity at the beginning of the working section was ≅30 m/sec.

The variable (negative and then positive) downstream temperature gradient was created by means of a strap (insert) placed on the upper wall of the working section of the tunnel. The strap had a length of 850 mm and was made in the form of two adjoining wedges with a rounding-off in the region of the "joint." To create pre-separation flow conditions, on the lower wall of the working section projections-steps of height $S$ = 60 and 90 mm were set. The afterbody of each projection ended with a wedge. The stable position of the line of transition was provided by a vortex generator in the form of a strip of emery paper pasted on the undersurface at beginning of the working section.

The mean value of the longitudinal velocity $u(y)$ was measured by a pneumometric technique. An impact pressure microhead of thickness 0.4 mm (inside diameter 0.2 mm, length along the transverse coordinate 1.0 mm) was used. The static pressure was measured with the help of drainage holes. In a number of experiments, we used a static pressure head moving near the undersurface of the working section. The recommended in [14] corrections taking into account the influence of the velocity gradient and the nearness of the wall on the readings of the Pitot tube were not applied, since the estimates have shown that they are small. The influence of external turbulence on the mean velocity profile and on the friction coefficient, according to [15], did not exceed 2%.

**Test Data. Comparison of the Theoretical and Experimental Data.** The experimental values of the parameters $x$, $u_0$, $\mathrm{Re}_*$, $\delta^*$, $H$ and the calculated values of $c_\mathrm{f}$, $\omega$, $G$, $\beta$ are given in Table 1. The described method for calculating the mean velocity profiles and the turbulent friction coefficient is based on analytical approximations of the velocity defect profiles for two characteristic kinds of flow conditions corresponding to either relatively small (5) or large (close to the pre-separation conditions (9)) longitudinal pressure gradients. Experimental check of these "limiting" dependences is particularly important for (9) differing greatly from (8) proposed in [6]. Comparison to the experimental data of [10] given in Fig. 1 speaks in favor of (9). The relations obtained for such limiting conditions compared to those proposed in [6] are much simpler. In this case, neglecting (as in [6]) the logarithmic segment of the velocity profile, i.e., assuming $\omega$ = 0 and $N$ = 1, we have

$$\varphi = \varphi_2 + \varphi_3\,, \quad 0.5657\,\frac{u_\mathrm{s}}{u_0} = 1 - \frac{1}{H} + \frac{C_1}{\mathrm{Re}_*}\left(1 + \frac{1}{H}\right). \tag{28}$$

Thus, the procedure of calculating $u_\mathrm{s}/u_0$ becomes much simpler compared to [6], and the value of $c_\mathrm{f} = 2\varepsilon^2$ is calculated (in the first approximation) by the known value of $u_\mathrm{s}/u_0$ upon determining $\omega$ from the second relation of (23). At the separation point $u_\mathrm{s}/u_0 = 1$, and then from (28) the dependence of the form parameter $H$ on the $\mathrm{Re}_*$ number given in Fig. 2 follows. When $\mathrm{Re}_* \to \infty$, the value of $H \to 2.30$.

Note that corrections for the influence of the wall zone of the flow, strictly speaking, are justified only in the presence of a logarithmic segment of the velocity profile, since, according to [4], this segment disappears with approach to the separation, and the wall zone of influence of viscosity $\Delta y$ increases in order of magnitude and is estimated as $\Delta y/\delta^* = O(\mathrm{Re}_*^{-2/3})$.

Let us check the degree of accuracy of the calculation of $u_\mathrm{s}/u_0$ with the aid of (28) by comparing to the experimental results given in [6]. From the graphs [6] we have: at $H$ = 1.81 and 1.75 the values of $u_\mathrm{s}/u_0$ are equal to 0.80 and 0.75, respectively. The calculation with the aid of (28) (we assume $C_1$ equal to zero, since the Re value in [6] is not given) yields: $u_\mathrm{s}/u_0$ = 0.79 and 0.76.

One of the main problems of the tests performed was the choice of the experimental function $N = N(\omega)$, which should provide a fairly exact approximation of the velocity profile in conditions intermediate between those indicated above. The test data have shown that an increase in the longitudinal pressure gradient leads to a very sharp change from the functional dependence (5) to (9) in the range $0.3 < \omega < 0.5$. The approximation thereby is provided by the relation

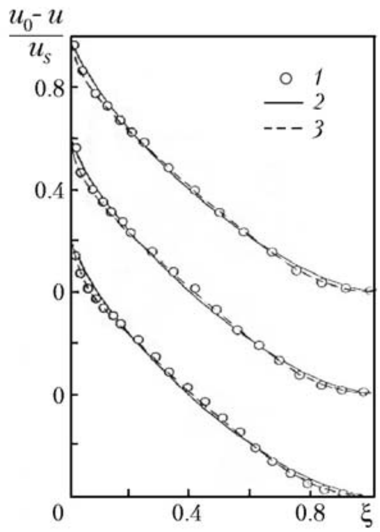
$$N = 0.4988\left[1 - 8\omega^3 + \sqrt{(1 - 8\omega^3)^2 + 0.01}\,\right]. \tag{29}$$

TABLE 1. Experimental Values of Parameters $x$, $u_0$, $\mathrm{Re}_*$, $\delta^*$, $H$ and Calculated Values of $c_f$, $\omega$, $G$, $\beta$ (bracketed are the results of the calculation at $C_1 = C_2 = 0$)

| $x$, mm | $u_0$, m/sec | $\mathrm{Re}_* \cdot 10^{-3}$ | $\delta^*$, mm | $H$ | $c_{fL-T} \cdot 10^3$ | $c_{fK} \cdot 10^3$ | $c_f \cdot 10^3$ | $\omega$ | $G$ | $\beta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| *Insert* | | | | | | | | | | |
| 0.20 | 31.1 | 6.23 | 2.96 | 1.36 | 3.06 | 3.19 | 3.07 (2.98) | 1.14 (0.91) | 6.56 (6.90) | –0.262 (–0.270) |
| 0.30 | 31.7 | 6.32 | 2.95 | 1.36 | 3.07 | 3.22 | 3.09 (3.00) | 1.23 (0.96) | 6.46 (6.80) | –0.358 (–0.368) |
| 0.40 | 32.3 | 6.16 | 2.86 | 1.34 | 3.15 | 3.32 | 3.17 (3.08) | 1.63 (1.18) | 6.18 (6.51) | –0.349 (–0.360) |
| 0.50 | 32.7 | 6.26 | 2.86 | 1.34 | 3.13 | 3.29 | 3.15 (3.06) | 1.53 (1.13) | 6.23 (6.57) | –0.239 (–0.246) |
| 0.62 | 32.9 | 6.48 | 2.96 | 1.34 | 3.10 | 3.22 | 3.12 (3.04) | 1.49 (1.12) | 6.26 (6.58) | 0.025 (0.026) |
| 0.70 | 32.9 | 6.96 | 3.18 | 1.35 | 3.03 | 3.11 | 3.04 (2.97) | 1.30 (1.03) | 6.40 (6.70) | 0.231 (0.237) |
| 0.78 | 32.4 | 7.57 | 3.50 | 1.36 | 2.92 | 2.96 | 2.92 (2.85) | 1.02 (0.86) | 6.71 (7.00) | 0.425 (0.436) |
| 0.90 | 31.8 | 8.56 | 4.07 | 1.37 | 2.77 | 2.78 | 2.77 (2.70) | 0.80 (0.71) | 7.12 (7.40) | 0.570 (0.584) |
| 1.00 | 31.2 | 9.37 | 4.53 | 1.38 | 2.66 | 2.66 | 2.65 (2.60) | 0.69 (0.62) | 7.45 (7.71) | 0.505 (0.516) |
| 1.10 | 30.5 | 9.92 | 4.86 | 1.39 | 2.61 | 2.62 | 2.59 (2.54) | 0.64 (0.58) | 7.61 (7.87) | 0.328 (0.335) |
| 1.20 | 30.1 | 10.2 | 5.05 | 1.38 | 2.62 | 2.64 | 2.62 (2.57) | 0.69 (0.62) | 7.44 (7.68) | 0.162 (0.165) |
| *Barrier, S = 60 mm* | | | | | | | | | | |
| 1.84 | 29.7 | 8.85 | 4.86 | 1.36 | 2.79 | 2.84 | 2.79 (2.73) | 0.90 (0.78) | 6.92 (7.19) | 0.066 (0.068) |
| 2.06 | 29.7 | 9.86 | 5.22 | 1.38 | 2.66 | 2.62 | 2.65 (2.60) | 0.72 (0.65) | 7.35 (7.60) | 0.155 (0.159) |
| 2.13 | 29.4 | 10.1 | 5.34 | 1.37 | 2.65 | 2.65 | 2.65 (2.60) | 0.73 (0.66) | 7.31 (7.55) | 0.509 (0.519) |
| 2.18 | 29.2 | 10.4 | 5.53 | 1.38 | 2.62 | 2.63 | 2.62(2.57) | 0.71 (0.65) | 7.38 (7.62) | 0.709 (0.723) |
| 2.23 | 29.2 | 11.7 | 6.24 | 1.38 | 2.51 | 2.47 | 2.50 (2.46) | 0.62 (0.57) | 7.2 (7.93) | 1.47 (1.50) |
| 2.28 | 28.4 | 11.9 | 6.49 | 1.40 | 2.43 | 2.12 | 2.41 (2.29) | 0.53 (0.45) | 8.15 (8.50) | 2.82 (2.96) |
| 2.33 | 27.4 | 13.5 | 7.62 | 1.45 | 2.21 | | 1.86 (1.80) | 0.28 (0.26) | 10.0 (10.3) | 7.05 (7.30) |
| 2.8 | 25.8 | 17.8 | 10.6 | 1.59 | 1.69 | | 1.09 (1.05) | 0.13 (0.12) | 15.7 (16.1) | 25.5 (26.4) |
| 2.40 | 24.8 | 22.2 | 13.9 | 1.76 | 1.25 | | 0.55 (0.53) | 0.065 (0.063) | 25.9 (26.6) | 69.6 (72.2) |
| 2.42 | 24.3 | 27.3 | 17.2 | 1.96 | 0.90 | | 0.21 (0.20) | 0.032 (0.031) | 47.8 (49.4) | 229 (242) |
| 2.43 | 24.0 | 31.7 | 20.2 | 2.14 | 0.66 | | 0.053 (0.047) | 0.014 (0.013) | 103 (109) | 1060 (1180) |
| *Barrier, S = 90 mm* | | | | | | | | | | |
| 0.94 | 29.7 | 6.7 | 3.47 | 1.39 | 2.88 | 2.93 | 2.87 (2.79) | 0.76 (0.65) | 7.23 (7.58) | 0.494 (0.509) |
| 1.00 | 29.3 | 7.3 | 3.82 | 1.41 | 2.75 | 2.78 | 2.73 (2.66) | 0.64 (0.56) | 7.64 (7.98) | 0.621 (0.638) |
| 1.10 | 28.4 | 8.33 | 4.48 | 1.43 | 2.58 | 2.56 | 2.54 (2.28) | 0.51 (0.38) | 8.27 (8.94) | 1.91 (2.13) |
| 1.20 | 25.9 | 12.1 | 7.14 | 1.58 | 1.89 | 1.74 | 1.25 (1.20) | 0.14 (0.13) | 14.5 (15.1) | 15.1 (15.8) |
| 1.25 | 23.9 | 18.6 | 11.9 | 1.91 | 1.06 | | 0.30 (0.28) | 0.041 (0.039) | 38.4 (40.0) | 156 (166) |

A direct comparison between the experimental and theoretical mean velocity profiles at $C_1 = -50.5$, $C_2 = -712$ is given in Fig. 3. In so doing, we used the experimental values of $H$, $\mathrm{Re}_*$ and relations (11), (12), (23), (24), (29). The $\delta$ value needed for calculating $y = \xi\delta$ was determined from relation (25). In Fig. 3, for flow conditions close to separation a marked difference between the experimental and calculated velocity profiles is observed (Fig. 3b). It is logical to explain this by the great difference of the flow from a plane-parallel one. From (26) the slope of the upper boundary of the boundary layer $d\delta/dx$ at $\delta = O(\theta)$ is estimated by the value of $\beta\varepsilon^2 = \beta c_f/2$. For the case of "equilibrium" separation, according to [4], $\beta\varepsilon^2 = 0.01$ and the influence of nonparallelism is insignificant (Fig. 1). However, as applied to profiles 5 and 6 in Fig. 3b, from the data of Table 1 we have, respectively, $\beta\varepsilon^2 = 0.02$ and 0.03.

The results of the calculation show that the velocity profiles corresponding to $C_1 = -50.5$, $C_2 = -712$, and $C_1 = C_2 = 0$ are very close, but for determining the drag coefficient at $\omega > 0.5$ the chosen nonzero values of these constants are preferable if we are going to use the popular Ludwieg–Tillman formula (20). At the same time, it should be remembered that when the velocity profile tends to a separation one ($\omega \to 0$) the proposed method gives a more correct result $c_f \to 0$ compared to formula (20) according to which $c_f \to 0$ when $H \to \infty$. The extent to which the testing conditions differ from equilibrium can be estimated by the data presented in Fig. 4, where the calculated values of the function $G$ (32) are shown versus $\beta$ ($C_f = -50.5$, $C_2 = -712$) for tests with an alternating pressure gradient. Note that the simple relation $G = G(\beta)$ for equilibrium flows is valid, strictly speaking, only in the limited case of



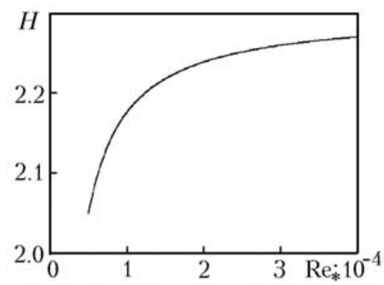




Fig. 1. Comparison of the calculated relations (8) and (9) to the experimental results of [10]. The three families of curves and points pertain respectively (from top to bottom) to the three cross-sections at $x$ = 3.532, 3.930, and 4.332 (for each family of curves and points "zero" of the ordinate axis is shifted): 1) experiment; 2) (8); 3) (9).

Fig. 2. Dependence of the form parameter $H$ on $\mathrm{Re}_*$ at the separation point.

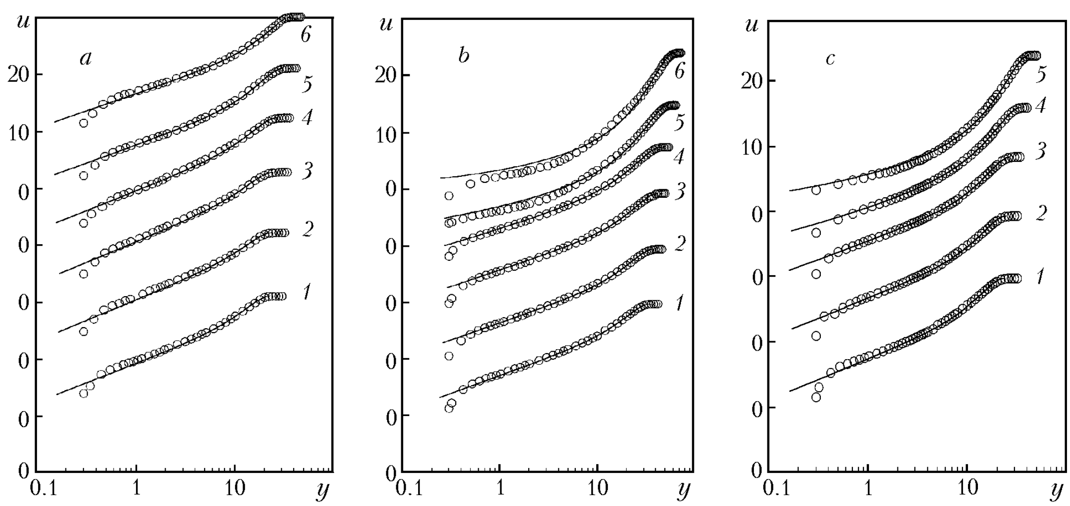



Fig. 3. Calculated (curves) and experimental (points) mean velocity profiles: a) flow without separation with an alternating pressure gradient [1) $x$ = 0.2 m, 2) 0.4, 3) 0.62, 4) 0.78, 5) 1.0, 6) 1.4]; b) flow with separation, barrier height $S$ = 60 mm [1) $x$ = 1.845 m, 2) 2.13, 3) 2.23, 4) 2.33, 5) 2.405; 6) 2.43); c) flow with separation, barrier height $S$ = 90 mm [1) $x$ = 0.945 m, 2) 1.005, 3) 1.105, 4) 1.205, 5) 1.255].

$\mathrm{Re}_* \to \infty$ when the terms with $C_1$ and $C_2$ can be neglected (this, in particular, can explain the difference between the experimental values of $G(\beta)$ obtained by different authors). Taking into account the foregoing, we used for comparison two relations proposed, respectively, in [16] and [17]:

$$G(\beta) = 6.1\,(\beta + 1.81)^{1/2} - 1.7\,, \tag{30}$$

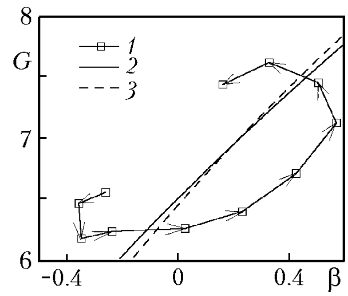



Fig. 4. Comparison of the equilibrium and nonequilibrium values of the function $G$: 1) nonequilibrium flow with an alternating pressure gradient (tests with an insert); 2) equilibrium flow of [16]; 3) equilibrium flow of [17].

$$G(\beta) = 5.77\,(\beta + 1.25)^{1/2}\,. \tag{31}$$

In determining $\beta$, the velocity gradient was found by approximating the experimental data presented in Table 1. For $\varepsilon = (c_f/2)^{1/2}$ and $\omega$, we chose the values at $C_1$, $C_2 \neq 0$ and calculated the parameter $G$ by the formula

$$G = -\frac{A}{\omega}\int_0^1 \varphi^2 d\xi \Big/ \int_0^1 \varphi d\xi\,. \tag{32}$$

**Conclusions.** As opposed to the widely used methods for calculating the turbulent boundary layer using the notion "turbulent viscosity" [18], the application of the proposed method requires the use of two more relations defining two certain parameters of the layer. One of them is the momentum conservation equation. For the special case of equilibrium (self-similar) flow, the second relation can be the experimental dependence $G = G(\beta)$. But in the general case one has to use for this purpose one more ordinary differential equation which should describe the tending of the boundary layer to equilibrium. In principle, such an approach was proposed in [19]. However, we would prefer a semiempirical derivation of this "relaxation" equation closing the proposed relations. Note that an attempt, though not quite successful, to derive such an equation has already been made in [16]. The "universal" dependences obtained there are important in themselves, describing fairly well the mean velocity profiles for flows without separation, as well as for flows with separation from a "smooth" surface.

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## NOTATION

$A$ = 2.44, $B$ = 5.0, constants in the "law of the wall" [9]; $C_1$, $C_2$, constants in (21); $c_f$, local friction coefficient in (1); $H = \delta^*/\theta$, form parameter; $h$, characteristic size of the logarithmic segment of the velocity profile, mm; $p$, pressure, N/m$^2$; $\mathrm{Re}_* = u_0\delta^*/\nu$; $\mathrm{Re}_\theta = u_0\theta/\nu$; $S$, barrier height, mm; $u$, longitudinal component of velocity, m/sec; $u_0$, velocity outside the boundary layer at $p = p_w$, m/sec; $x$, $y$, coordinates along the surface of the body and along its normal, m, mm; $y_{\mathrm{lin}}$, upper boundary of the wall linear section of the velocity profile, mm; $y^+_{\mathrm{lin}} = \varepsilon u_0 y_{\mathrm{lin}}/\nu$; $\delta^*$, $\theta$, $\delta$, displacement thickness, momentum thickness, and conditional thickness of the boundary layer, mm; $\nu$, kinematic viscosity, m$^2$/sec; $\Pi$, Coles parameter in (2); $\rho$, density, kg/m$^3$; $\tau_w$, friction stress on the body surface, N/m$^2$; $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi$, functions in (11); $\omega$, parameter of the form of the velocity defect profile in (10), (29). Subscripts: f, friction; K, Clauser; L–T, Ludwieg–Tillman; lin, linear; s, scale; 0, outside the boundary layer; w, wall.

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